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Sommersemester 2019

Albert-Ludwigs-Universität Freiburg

Finite Simple Groups

Exercise Sheet 7

Due 02.07.2019

Exercise 1 (4 Points).
Let G be $\mathrm{SL}(2,2^k)$ and let t be an element of order 2. Prove that $C_G(t)$ is isomorphic to $C_2 \times \overset{k}{\cdots} \times C_2$.

Exercise 2 (4 Points).
Let F be the finite field of order q, where q is an odd prime power.

1. Show that elements of $\mathrm{SL}(2,q)$ of the form
$$\begin{pmatrix} a & b \\ -(1+a^2)b^{-1} & -a \end{pmatrix},$$
with $b \in F^\times$ and $a \in F$, yield elements of order 2 of $\mathrm{PSL}(2,q)$.
2. Conclude that $\mathrm{PSL}(2,q)$ cannot be embedded into $\mathrm{SL}(2,q)$.

Exercise 3 (6 Points).
The extended centralizer of an element x of a group G is defined as
$$C_G^*(x) = \{g \in G : x^g \in \{x, x^{-1}\}\}.$$

1. Prove that $C_G^*(x)$ is a subgroup of G and that $C_G(x) = C_G^*(x)$ whenever x and x^{-1} are not conjugate or $x^2 = 1$.
2. Show that the set of elements $g \in G$ such that $x^g = x^{-1}$ is empty or a coset of $C_G(x)$.
3. Conclude that $|C_G^*(x) : C_G(x)| = 2$ whenever $x^2 \neq 1$ and x and x^{-1} are conjugate.

Exercise 4 (6 Points).
Recall that the finite dihedral group D_{2n} is the semidirect product of a cyclic group $\langle a \rangle$ of order n and a cyclic group $\langle t \rangle$ of order 2 such that $a^t = a^{-1}$. See Exercise 3, Exercise Sheet 4.

1. Assume that D is a finite non-abelian group geneterated by two elements x and y such that x has order 2 and $y^x = y^{-1}$. Prove that D is a dihedral group D_{2n} for some n.
2. Prove that every finite dihedral group can be generated by two involutions.
3. Prove that if a finite group is generated by two involutions, then it is a dihedral group.